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AN ADDITIVITY PRINCIPLE FOR GOLDIE RANK[†]

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ABSTRACT

Let A be a noetherian ring. In general A will not admit a classical Artinian ring of quotients. Yet a problem in enveloping algebras leads one to consider the possible embedding of A in a prime ring B which is finitely generated as a left and a right A module. Under certain additional technical assumptions, it is shown that the set S of regular elements of A is regular in B and is an Ore set in both A and B with $S^{-1}A$ and $S^{-1}B$ Artinian. This enables one to establish the following additivity principle for Goldie rank. Let $\{P_1, P_2, \cdots, P_r\}$ be the set of minimal primes of A. Then under the above conditions it is shown that there exist positive integers $z_1, z_2, \cdots, z_r$ such that

$$\sum_{i=1}^{r} z_i \operatorname{rk}(A/P_i) = \operatorname{rk} B,$$

where rk denotes Goldie rank. This applies to the study of primitive ideals in the enveloping algebra of a complex semisimple Lie algebra.

1. Introduction

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $U(\mathfrak{g})$ its enveloping algebra, $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$, Prim $U(\mathfrak{g})$ the primitive spectrum of $U(\mathfrak{g})$ and $\pi: I \mapsto I \cap Z(\mathfrak{g})$ the projection of Prim $U(\mathfrak{g})$ onto Max $Z(\mathfrak{g})$. The problem of classifying Prim $U(\mathfrak{g})$ and in particular the Jantzen conjecture [1], 5.9, motivate [6], 11.1 an additivity principle for Goldie rank.[††] Our main result, Theorem 3.9, establishes such a principle in a form suitable for application to enveloping algebras. Indeed

[†] This paper was written while the authors were guests of the Institute for Advanced Studies, The Hebrew University of Jerusalem. The first author was on leave of absence from the Centre Nationale de la Recherche Scientifique, France.

[††] Also referred to as Goldie dimension, though the former terminology is more common in enveloping algebras.

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when suitably combined with the methods of [6], it gives [7] a strong lower bound on the cardinality of each Prim $U(\mathfrak{g})$ fibre relative to π. For $\mathfrak{g}$ simple of type A_n (Cartan notation) this coincides with Duflo's upper bound [4], prop. 9, and establishes Jantzen's conjecture in the form described in [6], 10.3.

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2. Gelfand–Kirillov dimension and Artinian rings of quotients

2.1. Let F be a commutative field and A a finitely generated F-algebra with identity 1. Fix a finite dimensional generating subspace V of A. For each $k \in \mathbf{N}^+$, let V^k denote the subspace of A spanned by the monomials $v_1 v_2 \cdots v_k : v_i \in V$ and define a filtration $A^0 \subset A^1 \subset A^2 \subset \cdots$, on A through $A^0 = F$, $A^k = V + V^2 + \cdots + V^k$. Given M a finitely generated left A module, let M^0 denote a finite dimensional generating subspace and define a filtration $M^0 \subset M^1 \subset M^2 \subset \cdots$, on M through $M^k = A^k M^0$. Define the *left Gelfand–Kirillov dimension* $d_A(M)$ of M through

$$d_A(M) := \overline{\lim_{k \to \infty}} \frac{\log \dim_F M^k}{\log k}.$$

(In particular $\mathrm{d}_A(M) = -\infty$ if $M = 0$.)

It is elementary and well-known that $\mathrm{d}_A(M)$ does not depend on the choice of generating subspaces V, M^0, as is also the following

LEMMA. *Let $M, M_1, M_2, \cdots, M_r$ be finitely generated left A modules, B an F-algebra containing A and finitely generated as a left A module. Then*

(i) $\mathrm{d}_A(M) = \sup_i \mathrm{d}_A(M_i)$, *given* $M = M_1 + M_2 + \cdots + M_r$.

(ii) $\mathrm{d}_A(A) \geqq \mathrm{d}_A(M)$, *for A considered as a left A module.*

(iii) $\mathrm{d}_A(M) \geqq \mathrm{d}_A(M_i)$ *for any subquotient M_i of M.*

(iv) $\mathrm{d}_A(L) \geqq \mathrm{d}_A(Lb)$ *for any $b \in B$ and any finitely generated A module L of B. Equality holds if $xb = 0 : x \in L$ implies $x = 0$.*

(v) *If I is an ideal of A contained in* $\mathrm{Ann}\, M$, *then* $\mathrm{d}_{A/I}(M) = \mathrm{d}_A(M)$.

If M is a right A module we can similarly define the *right Gelfand–Kirillov dimension* $\mathrm{d}'_A(M)$ of M. As A will be fixed throughout and because of (v) we shall generally drop the subscripts.

For any F-algebra C and any subset T of C we let $l_C(T)$ (resp. $r_C(T)$) denote the left (resp. right) annihilator of T in C. For the F-algebra A we drop the subscript.

2.2. Define A, B as in 2.1 and from now on assume that A is (left and right) Noetherian and that B is finitely generated as a left and right A module. Call B *left smooth* if $d_A(B) = d_A(L)$ for every non-zero left ideal L of B. Since B is left Noetherian it is enough by 2.1 (i) to assume this property for two-sided ideals. We may similarly define B to be right smooth.

LEMMA. *Suppose B is left smooth. Let I be an ideal of B. Then*
(i) *$B/l_B(I)$ is left smooth.*
(ii) *If $I \neq 0$, then* $d(B/l_B(I)) = d(B)$.
(iii) *If K is a left ideal of B satisfying* $d(B/K) < d(B)$, *then K is essential.*

Since B is right Noetherian we can write

$$I = \sum_{i=1}^{n} a_i B : a_i \in I.$$

Then $l_B(I) = \cap\, l_B(a_i)$ and the map $x + l(I) \mapsto (xa_1, xa_2, \cdots, xa_n)$ of $B/l(I)$ into $(Ba_1, Ba_2, \cdots, Ba_n)$ is injective. Now if $L + l(I)$ is a non-zero left ideal of $B/l(I)$, then $La_i \neq 0$ for some i and is a left ideal of B. This by 2.1 (i), (iii) and smoothness gives: $d(B/l(I)) \geqq d(L/l(I)) \geqq d(La_i) = d(B) \geqq d(B/l(I))$. Hence (i), (ii).

If $K \cap L = 0$, for some left ideal L of B, then L embeds in B/K and so by 2.1 (iii): $d(L) \leqq d(B/K) < d(B)$. Smoothness implies $L = 0$ and so gives (iii).

2.3 LEMMA. *Let B be a prime ring. Then B is left smooth.*

Let L be a non-zero left ideal of B. Since B is prime Noetherian, there exist $a_1, a_2, \cdots, a_n \in B$ such that $K := \Sigma La_i$ is an essential left ideal of B. By 2.1 (i), (ii), (iv) one has $d(K) = \sup d(La_i) \leqq d(L) \leqq d(B)$, so it is enough to prove the assertion for L essential. Choose $a \in L$ such that $l_B(a) = 0$. Then by 2.1 (iii), (iv): $d(L) \geqq d(Ba) = d(B)$, as required.

REMARK. The assertion clearly fails for semiprime rings.

2.4. From now on let $\{P_1, P_2, \cdots, P_r\}$ denote the set of minimal primes of A and $N := \cap\, P_i$ the nilradical of A. One has $N^s = 0$ for some positive integer s.

LEMMA. *Let P be a minimal prime of A. Then there exist ideals I, J of A such that $IJ \neq 0$ and $IPJ = 0$.*

Observe that $(P_1 P_2 \cdots P_r)^s = 0$ and choose a product of the P_i of minimal length which vanishes. Each P_i must occur in this product for otherwise $P_k \subset P_j$ for some $k \neq j$. From this one obtains a suitable choice of I, J.

REMARKS. This is a weak form of an unpublished result of Kaplansky which further asserts that we can choose I, J so that $P = \{a \in A : IaJ = 0\}$. Note also that there is always at least one minimal prime P for which $l(P) \neq 0$.

2.5 LEMMA. *Suppose* $a \in A$ *satisfies* $l(a) = 0$. *Then*

(i) $\mathrm{d}(A/Aa) \leqq \mathrm{d}(A) - 1$.

(ii) *If* A *is left smooth and* $\mathrm{d}(A) < \infty$, *then* a *is regular.*

The proof of (i) exactly follows that of [2], 3.4.

(ii) Choose $b \in r(a)$. Then Ab is isomorphic to a subquotient of A/Aa so by (i) and 2.1 (iii) we obtain $\mathrm{d}(Ab) \leqq \mathrm{d}(A/Aa) < \mathrm{d}(A)$. By left smoothness this gives $Ab = 0$ and hence $r(a) = 0$, as required.

2.6. From now on we shall assume that $\mathrm{d}(A) < \infty$. Suppose that A is left smooth. Call A *bi-smooth* if $\mathrm{d}(I/J) = \mathrm{d}'(I/J)$ for all ideals I, J: $\supset J$ of A. This "two-sided" property trivially translates to quotients and so by 2.2 (i) if A is bi-smooth, then so is $A/l(I)$, for any ideal I. We shall see in 3.1 that this two-sided property holds for the enveloping algebra of a finite dimensional Lie algebra. Again from 2.2 we see that a bi-smooth ring is right smooth.

PROPOSITION. *Suppose* A *is bi-smooth. Then*

(i) $\mathrm{d}(A/P) = \mathrm{d}(A)$, *for each minimal prime* P *of* A.

(ii) *If* L *is a left ideal of* A *satisfying* $\mathrm{d}(A/L) < \mathrm{d}(A)$, *then* $L/(L \cap N)$ *contains a regular element of* A/N.

(iii) *Suppose* $N \neq 0$. *If* P' *is a minimal prime of* $A/l(N)$, *then its inverse image* Q *is a minimal prime of* A.

(i) Pick I, J as in the conclusion of 2.4. Let $\bar{I}$ (resp. $\bar{P}$) denote the image of I (resp. P) in $A/l(J)$. Then $\bar{I}\bar{P} = 0$ and so $\bar{I}$ considered as a finitely generated right $\bar{A}/\bar{P}$ module satisfies $\mathrm{d}'(\bar{I}) \leqq \mathrm{d}'(\bar{A}/\bar{P}) \leqq \mathrm{d}'(A/P)$, by 2.1 (ii). Now $IJ \neq 0$, so $\bar{I} \neq 0$ and $J \neq 0$ and since A is left smooth, 2.2 (i), (ii) gives $\mathrm{d}(\bar{I}) = \mathrm{d}(A/l(J)) = \mathrm{d}(A)$. By the two-sidedness property, this and our previous inequality gives (i).

(ii) Let P be a minimal prime of A and let $\bar{L}$ denote the image of L in A/P. Since $\mathrm{d}(\bar{A}/\bar{L}) \leqq \mathrm{d}(A/L) < \mathrm{d}(A)$ by hypothesis, it follows from (i), 2.2 (iii) and 2.3 that $\bar{L}$ is an essential left ideal of the prime ring A/P. This gives $r(L) \subset P$ and so $r(L) \subset N$. Hence $L/(L \cap N)$ is essential in A/N and so contains a regular element of the semiprime ring A/N.

(iii) Q is a prime ideal of A. Suppose it contains strictly some minimal prime

P of A. Then Q/P contains a regular element a of A/P and so by 2.1 (iii), 2.5 (i) one obtains $\mathrm{d}(A/Q) \leqq \mathrm{d}(A/Aa) < \mathrm{d}(A)$. Yet if $N \neq 0$, then by (i) and 2.2 (i), (ii): $\mathrm{d}(A/Q) = \mathrm{d}((A/l(N))/P') = \mathrm{d}(A/l(N)) = \mathrm{d}(A)$ and this contradiction proves (iii).

REMARK. By (iii) the radical $\sqrt{l(N)}$ of $l(N)$ is an intersection of minimal primes of N and either

(a) $A/l(N)$ has strictly less minimal primes than A, or

(b) $l(N) \subset N$ and the index of nilpotence of the nilradical of $A/l(N)$ is strictly less than that of N.

2.7. From now on let S denote the set of regular elements of A.

THEOREM. *Let A be a bi-smooth Noetherian F-algebra satisfying* $\mathrm{d}(A) < \infty$. *Then S is an Ore set in A and the ring of quotients $S^{-1}A$ is Artinian with nilradical S^{-1}.*

By [8] a Noetherian ring A admits a classical Artinian ring of quotients iff for each $a \in A$ with $a + N$ regular in A/N one has a regular in A. By the remark of 2.6 and the fact that $A/l(N)$ is bi-smooth we may prove this by induction on the number of minimal primes and on the index of nilpotence. Thus we are reduced to proving that A satisfies the conclusion of the theorem given that $A/l(N)$ does. Set

$$I: = \{n \in N: sn = 0 \text{ for some } s + l(N) \text{ regular in } A/l(N)\}.$$

Since N is a left $A/l(N)$ module, it follows from the left Ore condition in $A/l(N)$ that I is an ideal of A (contained in N). Now A is right Noetherian, so we may write

$$I = \sum_{i=1}^{m} n_i A : n_i \in I,$$

and then again by the left Ore condition in $A/l(N)$ there exists some $t + l(N)$ regular in $A/l(N)$ such that $tI = 0$. Then by 2.1 (iii), 2.5 (i) it follows that

$$\mathrm{d}(A/l(I)) \leqq \mathrm{d}((A/l(N))/(At + l(N))/l(N)) < \mathrm{d}(A/l(N)) \leqq \mathrm{d}(A).$$

Then by 2.2 (ii), $I = 0$. Suppose $a + N$ is regular in A/N. Then $r(A) \subset N$. Furthermore $a + \sqrt{l(N)}$ is regular in $A/\sqrt{l(N)}$ and so $a + l(N)$ is regular in $A/l(N)$, by the induction hypothesis. Hence $r(a) = 0$ and so a is regular by 2.5 (ii), with left replaced by right.

2.8. Retain the notation and hypotheses of 2.7. By 2.6 (ii)

COROLLARY. *Let L be a left ideal of A satisfying* $d(A/L) < d(A)$, *then* $L \cap S \neq \varnothing$.

3. The additivity principle

3.1. Let $\mathfrak{a}$ be a finite dimensional F-Lie algebra and let $U(\mathfrak{a})$ denote its enveloping algebra. In this section we take A to be a quotient of $U(\mathfrak{a})$. Then A admits an identity 1, is Noetherian, finitely generated and by 2.1 (iii), [2], 5.4: $d(A) \leqq d(U(\mathfrak{a})) = \dim \mathfrak{a} < \infty$. We let B denote an F-algebra containing A as a subalgebra so that 1 is also the identity of B. Through the principal antiautomorphism, $U(\mathfrak{a})$ is isomorphic to its opposite algebra and we note that B is defined as a left and as a right $U(\mathfrak{a})$ module (and hence as a $U := U(\mathfrak{a}) \otimes U(\mathfrak{a})$ module). Given $X \in \mathfrak{a}$, $b \in B$, write $(\operatorname{ad} X)b := Xb - bX$ and let $(\operatorname{ad}\mathfrak{a})b$ denote the F subspace of B generated by the $(\operatorname{ad}X)b$: $X \in \mathfrak{a}$. Call $b \in B$, *locally* $\operatorname{ad}\mathfrak{a}$ *finite* if

$$\dim_F\left(\sum_{t=0}^{\infty} (\operatorname{ad}\mathfrak{a})^t b\right) < \infty.$$

Obviously each $a \in A$ is locally $\operatorname{ad}\mathfrak{a}$ finite. We shall assume (and this imposes a strong condition on A) that B satisfies

(a) Every $b \in B$ is locally ad $\mathfrak{a}$ finite.

(b) B is finitely generated as a U module.

(c) B is a prime ring.

As noted in [6], 2.3, it follows from (a), (b) that B is finitely generated both as a left and as a right $U(\mathfrak{a})$ module and is hence a Noetherian ring. Again by [6], 2.3, we have the

LEMMA. *Consider B as a U module. If V is a subquotient of B then considered respectively as a left and as a right $U(\mathfrak{a})$ module it satisfies* $d(V) = d'(V)$.

REMARK. The U submodules of A are just its ideals.

EXAMPLE. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{p}$ a parabolic sub-algebra of $\mathfrak{g}$ and V a finite dimensional simple $U(\mathfrak{p})$ module. Let M denote the $U(\mathfrak{g})$ module induced from V and set $I = \operatorname{Ann} M$, $A = U(\mathfrak{g})/I$. Consider $\operatorname{Hom}_C(M, M)$ as a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ module [6], 3.2, and let B denote the subalgebra of $\operatorname{Hom}_C(M, M)$ of all $\operatorname{ad}\mathfrak{g}$ finite elements. Then B is a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ submodule of $\operatorname{Hom}_C(M, M)$ containing A as a subalgebra. By [6], 4.3 (i), B is finitely generated as a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ module and by [6], 4.8 (ii) it is a prime ring. Finally $\operatorname{rk} B = \dim V$, [6], 5.10.

3.2 LEMMA. *Suppose* $b \in B$ *satifies* $l_B(b)=0$ (*resp.* $r_B(b)=0$). *Then* $d(B/Bb) \leqq d(B)-1$ (*resp.* $d'(B/bB) \leqq d'(B)-1$).

Both parts are similar and we consider only the first. Let B^0 be a finite dimensional generating subspace for B considered as a left A module. Then $B^0 b$ is a finite dimensional generating subspace for Bb and we can choose $t \in \mathbf{N}$ such that $B^0 b \subset A^t B^0$. Then for all integers $k \geqq t$ we have $(A^{k-t}B^0)b \subset A^k B^0$ and since $l_B(b)=0$, one has $\dim(A^{k-t}B^0)b = \dim(A^{k-t}B^0)$. Set $f(k)=\dim A^k B^0$. Then

$$d(B) = \overline{\lim_{k\to\infty}} \frac{\log f(k)}{\log k},$$

whereas by our first observation

$$d(B/Bb) \leqq \overline{\lim_{k\to\infty}} \frac{\log(f(k)-f(k-t))}{\log k}.$$

Now for k sufficiently large, $f(k)$ is polynomial in k (the Hilbert–Samuel polynomial, see [6], 2.1, for example) and $\deg f = d(B)$. If $\deg f \geqq 1$, then $d(B/Bb) \leqq \deg f - 1$, as required. Otherwise B must be finite dimensional over F and so $B = Bb$, which establishes the assertion in this case.

3.3 LEMMA. *Let* M *be a left* $U(\mathfrak{a})$ *submodule of* A. *Then*

(i) $d(MU(\mathfrak{a})) = d(A)$.

(ii) $d(MB) = d(M)$.

Let V denote the map of $\mathfrak{a} \otimes A$ in $U(\mathfrak{a})$ and M^0 a finite dimensional generating subspace for M. By hypothesis 3.1 (a) there exists $t \in \mathbf{N}$ such that

$$\sum_{k=0}^{\infty} (\operatorname{ad}\mathfrak{a})^k M^0 = \sum_{k=0}^{t} (\operatorname{ad}\mathfrak{a})^k M^0.$$

Since M is a left $U(\mathfrak{a})$ module it follows that $MU(\mathfrak{a}) = MV^t$ and so (i) follows from 2.1 (i), (iv). Since B is finitely generated as a left $U(\mathfrak{a})$ module, (ii) similarly follows from (i).

3.4. Recall (hypothesis 3.1 (c)) that B is assumed prime and is a Noetherian ring.

LEMMA. *If* $b \in B$ *satisfies* $d'(B/r_B(b)) < d'(B)$, *then* $b = 0$.

Obviously 2.2, 2.3 hold with left replaced by right. Then by 2.3 it follows that B is right smooth and by 2.2 (iii) that $r_B(b)$ is an essential right ideal of B and so contains a regular element. This gives $b = 0$, as required.

3.5 Proposition. *Fix $a \in A$. The following three assertions are equivalent:*
(i) $l(a) = 0$. *(Recall that* $l(a) = l_A(a)$.*)*
(ii) $r(a) = 0$.
(iii) *a is regular in B.*

Since B is prime Noetherian it suffices to show that $l(a) = 0$ implies $r_B(a) = 0$. Choose $b \in B$ such that $ab = 0$. Then $Ab = A/l(b)$ up to an isomorphism of left A modules and so $\mathrm{d}(Ab) = \mathrm{d}(A/l(b)) \leqq \mathrm{d}(A/Aa) < \mathrm{d}(A)$ by the hypothesis, 2.1 (iii), 2.5 (i). Taking $M = Ab$ in 3.3 gives $\mathrm{d}(Ab) = \mathrm{d}(AbB) = \mathrm{d}'(AbB) \geqq \mathrm{d}'(bB)$, by 3.1 and 2.1 (iii). Yet $bB = B/r_B(b)$, up to an isomorphism of right B modules and so $\mathrm{d}'(B/r(b)) < \mathrm{d}(A) \leqq \mathrm{d}'(B)$ by 3.1 and 2.1 (iii). From 3.4 this gives $b = 0$, as required.

Remark. The proposition fails if $1 \in A$ is not the identity of B. This was needed for the isomorphism $Ab = A/l(b)$.

3.6 Lemma. (i) *A is bi-smooth.*
(ii) *Let K be an essential left (or right) ideal of B.*
Then $L := A \cap K$ contains a regular element of A.

(i) By 3.1 it suffices to prove that A is left smooth. Recalling that B is prime this follows from 3.3 (ii) and 2.3.

(ii) Since B is a prime ring it follows that K admits a regular element b of B. Then from 2.1 (iii), 3.2 and 3.3: $\mathrm{d}(A/L) \leqq \mathrm{d}(B/K) < \mathrm{d}(B) = \mathrm{d}(A)$. Combined with 2.8, this gives (ii).

3.7. Let S denote the set of regular elements of A and Fract B the classical ring of quotients of B (which is simple and Artinian). By 3.5, S is contained in the set of regular elements of B.

Corollary. (i) *S is an Ore set in A and $S^{-1}A$ is Artinian.*
(ii) *S is an Ore set in B and* $S^{-1}B = \mathrm{Fract}\, B$.

By 2.7 and 3.6 (i) it remains to prove (ii). Consider for example the left Ore condition. Given $s \in S$, $b \in B$, set $K = \{c \in B : cb \in Bs\}$. Since s is also regular in B a standard argument shows that K is an essential left ideal of B. Then by 3.6 (ii), we obtain $K \cap S \neq \varnothing$ and this establishes the first part of (ii). Finally given t regular in B, then Bt is an essential left ideal of B and so by 3.6 (ii) there exists $b \in B$ such that $s := bt \in A$. Hence t is invertible in $S^{-1}B$ which proves the second part of (ii).

3.8. Given $\mathcal{B}$ a prime Noetherian ring we recall that the Goldie rank rk $\mathcal{B}$ of

$\mathscr{B}$ is just the maximum number of direct summands of left (or right) ideals of $\mathscr{B}$. Given K a left $\mathscr{B}$ module we let rk K denote the maximum number of direct summands of left $\mathscr{B}$ submodules of K. Given $e \in \mathscr{B}$ a projection we remark that $e\mathscr{B}e$ is a prime Noetherian ring and rk $e\mathscr{B}e$ = rk $\mathscr{B}e$. Let $\mathscr{A}$ be a Artinian subring of $\mathscr{B}$ with identity 1 and assume that 1 is also the identity of $\mathscr{B}$. Let $\{Q_1, Q_2, \cdots, Q_r\}$ denote the set of (minimal) primes of $\mathscr{A}$ and $\mathscr{N}$ its nilradical. We may write $Q_i/\mathscr{N} = \mathscr{A}/\mathscr{N}\ (\bar{1} - \bar{x}_i)$: $i = 1, 2, \cdots r$, where $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_r)$ is the maximal set of pairwise orthogonal minimal central projections of $\bar{\mathscr{A}} := \mathscr{A}/\mathscr{N}$. By [5], sect. 8, prop. 5, there exists a set $\{e_1, e_2, \cdots, e_r\}$ of pairwise orthogonal projections of $\mathscr{A}$ satisfying $e_i - x_i \in \mathscr{N}$ for all i, and $\Sigma e_i = 1$.

LEMMA. *For all* $i = 1, 2, \cdots, r$

(i) $z_i := (rk\mathscr{B}e_i)/rk(\mathscr{A}/Q_i) \in \mathbf{N}^+$.

(ii) $\Sigma_{i=1}^r z_i \operatorname{rk}(\mathscr{A}/Q_i) = \operatorname{rk} \mathscr{B}$.

From $\mathscr{B} = \bigoplus \mathscr{B}e$ we obtain (ii). For (i) note that $e_i\mathscr{A}e_i$ has nilradical $\mathscr{N}e_i$ and is an Artinian subring of the prime ring $e_i\mathscr{B}e_i$ with identity e_i. Then through the algebra isomorphisms $(e_i\mathscr{A}e_i)/\mathscr{N}e_i = (\mathscr{A}/\mathscr{N})x_i = \mathscr{A}/Q_i$ it suffices to establish (i) in the case $r = 1$.

Let $\{\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_s\}$ be a maximal set of pairwise orthogonal minimal projections of $\mathscr{A}/\mathscr{N}$. By [5], sect. 8, prop. 5 there exists a set $\{f_1, f_2, \cdots, f_s\}$ of pairwise orthogonal projections of $\mathscr{A}$ satisfying $f_i - y_i \in \mathscr{N}$ for all i, and $\Sigma f_i = 1$. Yet $s = \operatorname{rk} \mathscr{A}/\mathscr{N}$ and the left $\mathscr{A}/\mathscr{N}$ modules $(\mathscr{A}/\mathscr{N})y_i : i = 1, 2, \cdots, s$ are pairwise isomorphic. By [5], sect. 8, prop. 1, the $\mathscr{A}f_i : i = 1, 2, \cdots, s$ are pairwise isomorphic left $\mathscr{A}$ modules. Since $\mathscr{A}$ is a subring of $\mathscr{B}$ it follows from [5]. sect. 7, prop. 4, that the $\mathscr{B}f_i : i = 1, 2, \cdots, s$ are pairwise isomorphic left $\mathscr{B}$ modules. From $\mathscr{B} = \bigoplus \mathscr{B}f_i$ we then obtain

$$\operatorname{rk} \mathscr{B} = \sum_{i=1}^{s} \operatorname{rk}(\mathscr{B}f_i) = s\,(\operatorname{rk} \mathscr{B}f_1),$$

which establishes (i).

3.9 THEOREM. *Let A be a quotient of the enveloping algebra* $U(\mathfrak{a})$ *of a finite dimensional F-Lie algebra* $\mathfrak{a}$. *Assume that A embeds in a prime ring B with identity* $1 \in A$ *which is a finitely generated* $U(\mathfrak{a}) \otimes U(\mathfrak{a})$ *module of locally* ad $\mathfrak{a}$ *finite elements. Let* $\{P_1, P_2, \cdots, P_r\}$ *be the set of minimal primes of A. Then for all* $i = 1, 2, \cdots, r$

(i) $d(A/P_i) = d(A)$,

(ii) *there exist* $z_i \in \mathbf{N}^+$, *such that*

$$\sum_{i=1}^{r} z_i \operatorname{rk}(A/P_i) = \operatorname{rk} B.$$

(i) follows from 3.6 (i) and 2.6 (i).

(ii) By 2.7, 3.5–3.7, the set S of regular elements of A is contained in the set of regular elements of B and is an Ore subset for both A and B. Let N be the nilradical of A. By 2.7, $\mathcal{A} := S^{-1}A$ has nilradical $\mathcal{N} := S^{-1}N$ and is an Artinian subring of the prime ring $\mathcal{B} := S^{-1}B$ with identity $1 \in \mathcal{A}$. If P is a minimal prime of A then by 2.4 we have $P \cap S = \varnothing$. Conversely if Q is not a minimal prime of A, then as in the proof of 2.6 (iii) it follows that $d(A/Q) < d(A)$ and so $Q \cap S \neq \varnothing$ by 2.8. Then by [3], 2.10: $\{Q_i := S^{-1}P_i : i = 1, 2, \cdots, r\}$ is the set of primes of $\mathcal{A}$ and $\operatorname{Fract} A/P_i = \mathcal{A}/Q_i$, up to isomorphism. Then (ii) follows from 3.8.

REMARK. Applying (i) to the example of 3.1 establishes a result which has long been suspected. Namely for the annihilator I of an induced module the quotients of $U(\mathfrak{g})$ defined by the minimal primes containing I have all the same Gelfand–Kirillov dimension.

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